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Stochastic partial differential equations and applications to hydrodynamics

by

B. Øksendal

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STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS AND APPLICATIONS TO HYDRODYNAMICS

Bernt Øksendal
Dept. of Mathematics
University of Oslo
Box 1053, Blindern
N-0316 Oslo, NORWAY

CONTENTS

CHAPTER 0. INTRODUCTION

The purpose of these notes is to explain how white noise and related methods can be used to model and describe important random dynamical phenomena. Because of personal interest and activity so far emphasis will be put on stochastic partial differential equations arizing in hydrodynamics, but the methods we give are general and not at all restricted to such equations. In this survey we will concentrate on *multidimensional* white noise; its construction, methods and applications.

Since the purpose is to describe nature mathematically it is important to focus on modelling, an aspect which has been neglected by many authors. Important questions are:

1. What features are we trying to describe?
2. What kind of observations do we have as a basis for our model?
3. When the mathematical model is established, how do proceed mathematically to solve the equations?
4. What information can be deduced from the solutions? Does this information agree with the observations?

This presentation is based on joint works with several authors: J. Gjerde, H. Gjessing, H. Holden, T. Lindstrøm, O. Martio, J. Ubøe and T.-S. Zhang. The starting point for our work was a project supported by VISTA, a research cooperation between the Norwegian Academy of Science and Letters and Den Norske Stats Oljeselskap A.S. (Statoil). The purpose of this project is to describe mathematically the flow of oil, gas and water in porous rocks. Such multi-phase flow can in general be described by a system of nonlinear partial differential equations. These equations represent an enormous challenge even if all the parameters of the system were known. However, in almost all practical applications the permeability properties of the rock are essentially unknown, and this makes the description several levels harder. The lack of information makes it natural to introduce stochastic models, but in what way and in what sense stochastics should enter the equations is in no way canonical. This is where the modelling questions arize.

Having settled with a stochastic partial differential equation (SPDE) as our model, the next step is to perform rigorous matematical calculus on it to produce the solution. Here we would like not just a proof of the existence (and uniqueness), but also some probabilistic properties of the solution, so that we can use the solution to get more information and more insight about the physical situation we are trying to describe.

In the following we will first (in Chapter 1) discuss the basic equations and some observed features of fluid flow in porous media, together with some results based on deterministic methods from the theory of weighted Sobolev spaces and quasiconformal mappings.

Then in Chapter 2 we will review some of the terminology and theory necessary for the stochastic model and its calculus. In particular, we construct multidimensional white noise and establish some of its properties.

In Chapter 3 we give a general scheme for solving multidimensional white noise equations with Wick products and we illustrate the method with an example.

Then in Chapter 4 we return to some of the specific SPDE's related to fluid flow where some of the parameters are modelled by functionals of multidimensional white noise.

CHAPTER 1. FRACTAL BOUNDARIES IN MOVING BOUNDARY PROBLEMS

1a) The basic equations for multi-phase fluid flow in porous media. The moving boundary problem.

One of the basic equations describing one phase fluid flow in a porous medium is the *Darcy law*, which states that

$$\vec{q}_t(x) = -\frac{1}{\mu}K(x) \bigtriangledown p_t(x) \tag{1.1}$$

where $\vec{q}_t(x)$ is the (seepage) velocity of the fluid at the time t and at the point $x \in \mathbf{R}^d$, $p_t(x)$ is the pressure of the fluid at t and x (the gradient is taken w.r.t. x), μ is the viscosity of the fluid and $K(x) = [K_{ij}(x)]_{i,j=1}^d$ is the *permeability matrix* of the medium at x. For a general heterogenous medium the value of $K(x)$ will vary with x, often very irregularly. Moreover, in general the medium may be *anisotropic*, in the sense that the macroscopic flow properties depend on the direction of the flow. Element number i,j of $K(x)$, $K_{ij}(x)$, can be interpreted as the fluid velocity induced in direction i by a unit pressure gradient in direction j (setting $\mu = 1$). It is generally assumed (and it can also be deduced from more basic assumptions) that this permeability matrix $K(x)$ is *symmetric* and *positive* (= non-negative) *definit*. In the *isotropic* case we have $K(x) = k(x)I$ where I is the $d \times d$ identity matrix and $k(x) \geq 0$.

In the case of m-phase fluid flow it is customary to assume that a similar kind of Darcy law holds for the fluid of each phase of the fluid. Combined with the continuity equation this leads to equations of the form

$$\frac{\partial \theta_i}{\partial t} + div(M_i(x,\theta) \bigtriangledown p_t^{(i)}(x)) = \xi_t^{(i)}(x) \; ; \quad i = 1, 2, \cdots, m \tag{1.2}$$

where $\theta_i = \theta_i(t,x), p_t^{(i)}(x)$ is the saturation and pressure, respectively, of phase $i, \theta = (\theta_1, \cdots, \theta_m), \xi_t^{(i)}(x)$ is the source rate of phase i and the divergence is taken w.r.t. x.

Summing over i and putting $\theta = \sum_{i=1}^{m} \theta_i, \xi_t = \sum_{i=1}^{m} \xi_t^{(i)}$ and assuming that the pressure p is the same in all phases we get an equation of the form

$$\frac{\partial \theta}{\partial t} + div(N(x,\theta) \bigtriangledown p_t) = \xi_t(x) \tag{1.3}$$

If we - as a first approximation - ignore the dependence of θ in N, i.e. we put $N(x,\theta) = N(x)$, we get

$$\frac{\partial}{\partial t}\theta(t,x) + div(N(x) \bigtriangledown p_t(x)) = \xi_t(x) \tag{1.4}$$

If we assume that the fluid occupies the whole volume, then $\theta(t,x) \equiv 1$ and we get

$$div(N(x) \bigtriangledown p_t(x)) = \xi_t(x) \tag{1.5}$$

Combined with either specified values of $p_t(\cdot)$ at the boundary of the domain considered or with specified values of the flux $\bigtriangledown p_t(\cdot)$ at the boundary, we see that (1.5) represents a *linear, second order semi-elliptic boundary value problem of Dirichlet or Neumann type*, respectively.

Substituting the solution p of (1.5) into the original equation (1.2) we get equations of the form

$$\frac{\partial \theta_i}{\partial t} + div(P_i(x,\theta)) = \xi_t^{(i)}\,; \quad i = 1, 2, \cdot, m \tag{1.6}$$

which is a system of *nonlinear, first order partial differential equations in the unknowns* $\theta_1, \cdots, \theta_m$.

We see that this approximation procedure splits the original equation (1.2) into two separate systems of equations: The pressure equation (1.5) and the saturation equations (1.6).

We will return to (a special case of) (1.6) in Chapter 3. In the following let us take a closer look at the original equation (1.2) in the case when $m = 1$, i.e. there is only one phase of the fluid. Then, if we assume that $\mu = 1$ as before and also that the density of the fluid is set equal to 1, we get

$$\frac{\partial}{\partial t}\theta(t,x) - div(K(x) \bigtriangledown p_t(x)) = \xi_t(x) \tag{1.7}$$

where $K(x)$ is the permeability matrix.

As stated above the permeability matrix $K(x)$ often varies irregularly with x and therefore (and for other reasons) it is necessary to consider the weak (variational) form of (1.7), namely

$$\begin{aligned} &-\int\int \theta(t,x)\psi(x)\phi'(t)dxdt + \int\int [\bigtriangledown\psi(x)]^T K(x) \bigtriangledown p_t(x)\phi(t)dxdt \\ &= \int\int \xi_t(x)\psi(x)\phi(t)dxdt \end{aligned} \tag{1.8}$$

for all $\phi \in C_0^\infty(\mathbf{R}), \psi \in C_0^\infty(\mathbf{R}^d)$ (where dx, dt denotes Lebesgue measure on $\mathbf{R}^d$, $\mathbf{R}$, respectively). Now let us assume that $\theta(t,x)$ can only assume two values:

$$\theta(t,x) = \theta_0(x) > 0 \quad \text{(complete saturation at } x\text{)}$$

or

$$\theta(t,x) = 0 \quad \text{(the medium is dry at } x\text{)}$$

Put

$$D_t = \{x; \theta(t,x) = \theta_0(x)\} \quad \text{(the wet region)}$$

Then

$$(1.9)\qquad -\int\int \theta(t,x)\psi(x)\phi'(t)dxdt = \int \frac{\partial}{\partial t}(\int_{D_t} \theta_0(x)\psi(x)dx)\phi(t)dt$$

Letting $\phi(\cdot)$ approach $\chi_{[0,t]}(\cdot)$ and substituting (1.9) in (1.8), we get

$$(1.10)\qquad \begin{aligned} &\int_{D_t\setminus D_0} \theta_0(x)\psi(x)dx + \int [\nabla\psi(x)]^T K(x) \nabla u_t(x)dx \\ &\qquad = \int \psi(x)(\int_0^t \xi_s(x)ds)dx, \end{aligned}$$

where

$$(1.11)\qquad u_t(x) = \int_0^t p_s(x)ds \quad \text{(the Baiocchi transformation)}$$

Let us now assume that the (measurable), symmetric, positive definite matrix function $K(x)$ satisfies the requirement that there exist a *p-admissible weight* $m(x) \geq 0$ such that

$$(1.12)\qquad c_1 \cdot m(x)|y|^2 \leq y^T K(x)y \leq c_2 \cdot m(x)|y|^2$$

for all $y \in \mathbf{R}^d$, where $c_1, c_2 > 0$ are constants (see [HKM] for definition and properties of p-admissible weights). If $U \subset \mathbf{R}^d$ is open we let $H_0^1(U,m)$ denote the weighted Sobolev space consisting of the closure of $C_0^\infty(U)$ with respect to the norm

$$\|\psi\|_{H_0^1} = [\int_U (\psi^2(x) + |\nabla\psi(x)|^2)dx]^{\frac{1}{2}}$$

From now on we assume that there exists a bounded open set $U \subset \mathbf{R}^d$ such that for all times t considered, say $t \in [0,T]$, we have $D_t \subset U$.

Equations (1.10), (1.11) together with the assumptions

$$(1.13)\qquad u_t \geq 0 \text{ and } u_t(\cdot) \in H_0^1(D_t, m) \quad \forall t \geq 0$$

is called *(the weak formulation of) the moving boundary problem* for D_t and $p_t(x)$ (or $u_t(x)$).

This name is better understood if we consider the following *classical* (strong) formulation of the problem: Under sufficient smoothness conditions one can show that (1.10), (1.11), (1.13) lead to the 3 equations

$$(1.14)\qquad div(K(x)\nabla p_t(x)) = -\xi_t(x) \quad ; x \in D_t$$

$$(1.15)\qquad p_t(x) = 0 \qquad x \in \partial D_t$$

$$(1.16) \qquad \theta_0(x) \cdot \frac{d}{dt}(\partial D_t) = -N^T(x)K(x) \bigtriangledown p_t(x) \; ; \; x \in \partial D_t$$

where $N^T(x) = (N_1(x), \cdots, N_d(x))$ is the outer unit normal of D_t at $x \in \partial D_t$.

For more information we refer to [R], [MØ]. In contrast to the classical formulation, the weak version always has a unique solution (if (1.12) holds):

THEOREM 1 [R, Sec.2] Suppose $\xi_t(x) \geq 0$. Then there is a unique solution $u_t(x), D_t$ of the weak formulation (1.10), (1.11), (1.13) of the moving boundary problem.

1b) Physical experiments. The existence of porous media whose moving wet boundaries are fractal.

Since the permeability function $K(x)$ is so irregular, it may be tempting to replace it by its (constant) average $\overline{K}$ in the equations above. However, if K is constant, then it is well known ([Gu]), [BG]) that the boundary ∂D_t of the wet region will be smooth for all $t > 0$ if $\xi_t(x) > 0$. And this is far from what is being observed in physical experiments of fluid flow in porous rocks (see e.g. [LØU 3]). In fact, it has been conjectured that ∂D_t is a fractal for all $t > 0$ and that the Hausdorff dimension of ∂D_t, $dim_{\mathcal{H}}(\partial D_t)$, has the approximate values

$$(1.23) \qquad dim_{\mathcal{H}}(\partial D_t) \approx 1.7 \quad \text{if} \quad d = 2$$

$$(1.24) \qquad dim_{\mathcal{H}}(\partial D_t) \approx 2.5 \quad \text{if} \quad d = 3$$

See e.g. [MFJ].

Therefore, to be able to describe fluid flow in porous rocks realistically, it is necessary to be able to handle irregular permeability functions $K(x)$. A natural question is: If we allow $K(x)$ to be sufficienty irregular, can we deduce from the mathematical model above that ∂D_t is fractal?

This is a question about the connection between smoothness/irregularity properties of $K(x)$ and the corresponding moving boundary ∂D_t and this connection is so far not well understood. In fact, from private communications we have got the impression that several applied mathematicians doubt that *any* $K(x)$ - not matter how irregular - can produce a fractal moving boundary. In view of this the following result may be surprising to some:

THEOREM 2 [MØ]. For all dimensions d and all $\epsilon > 0$ there exists a porous medium (i.e. a permeability matrix $K(x)$ and a maximal saturation function $\theta_0(x) > 0$) such that if D_t solves the corresponding moving boundary problem (with D_0 equal to the unit ball centered at the origin and $\xi_t(x)$ being the unit point mass at the origin) then

$$(1.25) \qquad dim_{\mathcal{H}}(\partial D_t) > d - \epsilon$$

for at least one $t > 0$.

This medium is constructed as a distortion of a homogeneous medium by a suitable quasiconformal mapping. For details we refer to [MØ].

So ∂D_t is not only fractal, but its Hausdorff dimension can be made arbitrary close to d, for at least *one* $t > 0$. It is not clear for how many values of t this holds.

Note however, that in this theorem the medium (or $K(x)$) is *constructed* for the purpose of obtaining (1.25). The big question is: If we are given a physical medium, and hence an irregular $K(x)$, can we prove that the moving boundary is fractal (in some generic sense)? One way of making this question precise would be to introduce some kind of *stochastic* model for $K(x)$ and investigate the probability that the corresponding ∂D_t is fractal.

Such an approach is also useful from the point of view of other applications, because in many practical situations (for example in oil reservoirs) the values of $K(x)$ are virtually unknown and difficult to measure. This lack of information makes it natural to model $K(x)$ by some kind of *noise*. More precisely, we will represent $K(x)$ as a certain *functional of white noise*. By choosing the functional properly we can achieve that $K(x)$ has the right probabilistic properties. Moreover, we can benefit from the established theory and calculus of white noise analysis to solve the corresponding stochastic partial differential equations.

We will make this more precise in the next chapters. First we review briefly the necessary terminology and results from white noise theory.

CHAPTER 2. MULTIDIMENSIONAL WHITE NOISE, WICK PRODUCTS AND STOCHASTIC PERMEABILITY

The general reference for 1-dimensional white noise theory is [HKPS]. See also several of the contributions in this volume. Of particular interest to us here is the multidimensional white noise, which has been thoroughly discussed in [Gj]. We review some of the basic notations, terminology and results:

In the following we let $\mathcal{S}(\mathbf{R}^d)$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbf{R}^d$ with the usual topology and we let $\mathcal{S}'(\mathbf{R}^d)$ denote the dual space (the space of tempered distributions). Let $\mathcal{B}(\mathcal{S}'(\mathbf{R}^d))$ be the Borel subsets of $\mathcal{S}'(\mathbf{R}^d)$ in the weak star topology. If m is a natural number we put

$$\mathcal{S} := \prod_{i=1}^{m} \mathcal{S}(\mathbf{R}^d),\ \mathcal{S}' := \prod_{i=1}^{m} \mathcal{S}'(\mathbf{R}^d),\ \mathcal{B} := \prod_{i=1}^{m} \mathcal{B}(\mathcal{S}'(\mathbf{R}^d))$$

Since $\mathcal{S}$ is a countably Hilbert space [Gj] there exists by the Minlos theorem [HKPS] a probability measure μ on $\mathcal{B}$ such that

$$\int_{\mathcal{S}'} e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2} \quad \text{for all } \phi \in \mathcal{S} \tag{2.1}$$

Here $<\omega,\phi> = <\omega_1,\phi_1> + \cdots + <\omega_m,\phi_m>$ is the action of $\omega = (\omega_1,\cdots,\omega_m) \in \mathcal{S}'$ on $\phi = (\phi_1,\cdots,\phi_m) \in \mathcal{S}$ ($<\omega_k,\phi_k>$ being the action of $\omega_k \in \mathcal{S}'(\mathbf{R}^d)$ on $\phi_k \in \mathcal{S}(\mathbf{R}^d)$) and

$$\|\phi\| = \|\phi\|_{\mathcal{K}} = [\sum_{k=1}^{m} \|\phi_k\|^2_{L^2(\mathbf{R}^d)}]^{\frac{1}{2}} = [\sum_{k=1}^{m} \int_{\mathbf{R}^d} \phi_k^2(x)dx]^{\frac{1}{2}}$$

is the norm of ϕ in the Hilbert space $\mathcal{K} = \bigoplus_{k=1}^{m} L^2(\mathbf{R}^d)$ (the orthogonal sum).

One can show that (see e.g. [Gj])

$$\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_m,$$

where μ_k is the corresponding Minlos measure on $\mathcal{S}'(\mathbf{R}^d)$ (the 1-dimensional white noise probability measure).

We call $(\mathcal{S}', \mathcal{B}, \mu)$ *the multi-dimensional white noise probability space.*

The *m-dimensional white noise* $\vec{W} : \mathcal{S} \times \mathcal{S}' \to \mathbf{R}^m$ is defined by

$$\vec{W}_\phi(\omega) = (< \omega_1, \phi_1 >, \cdots, < \omega_m, \phi_m >) \in \mathbf{R}^m \tag{2.2}$$

if $\omega = (\omega_1, \cdots, \omega_m) \in \mathcal{S}, \phi = (\phi_1, \cdots, \phi_m) \in \mathcal{S}'$.

The connection to m-dimensional (d-parameter) Brownian motion $(B_x^{(1)}, \cdots, B_x^{(m)})$ is that

$$\vec{W}_\phi(\omega) = (\int \phi_1(x) dB_x^{(1)}, \cdots, \int \phi_m(x) dB_x^{(m)}); \ \phi \in \mathcal{S} \tag{2.3}$$

The *Hermite polynomials* $h_n(x)$ are defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \cdot \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}); \ n = 0, 1, 2, \cdots \tag{2.4}$$

and the *Hermite functions* $\xi_n(x)$ are defined by

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} h_{n-1}(\sqrt{2}x); \ n = 1, 2, \cdots \tag{2.5}$$

Let $\beta^{(j)} = (\beta_1^{(j)}, \beta_2^{(j)}, \cdots, \beta_d^{(j)})$ be multi-index nr. j in some fixed ordering of all d-dimensional multi-indices $\beta = (\beta_1, \cdots, \beta_d)$ and define

$$\eta_j = \xi_{\beta_1^{(j)}} \otimes \cdots \otimes \xi_{\beta_d^{(j)}} \quad ; \ j = 1, 2, \cdots \tag{2.6}$$

Then $\{\eta_j\}_{j=1}^\infty$ is an orthonormal basis of $L^2(\mathbf{R}^d)$. From this we can construct an orthonormal basis $\{e^{(n)}\}_{n=1}^\infty$ of $\mathcal{K}$ by choosing some linear ordering of the family of all m-vectors $\zeta_{ij} = (0, 0, \cdots, 0, \eta_j, 0, \cdots, 0)$ with η_j as coordinate nr. i, $1 \le i \le m, j = 1, 2, \cdots$.

To be specific let us put

$$e^{(n)} = \zeta_{ij} \tag{2.7}$$

where $i \in \{1, 2, \cdots, m\}$ and $j \in \{1, 2, \cdots\}$ are the unique integers such that

$$n = m(j-1) + i.$$

For $\alpha = (\alpha_1, \cdots, \alpha_n)$ a multi-index and $\omega \in \mathcal{S}'$ define

$$H_\alpha(\omega) := h_{\alpha_1}(< \omega, e^{(1)} >) h_{\alpha_2}(< \omega, e^{(2)} >) \cdots h_{\alpha_n}(< \omega, e^{(n)} >) \tag{2.8}$$

Then $\{H_\alpha\}_\alpha$ constitute an orthogonal basis for $L^2(\mu)$ and

(2.9) $$\|H_\alpha\|^2_{L^2(\mu)} = \alpha! := \alpha_1! \cdots \alpha_n!$$

(See [HKPS]). Therefore any $f \in L^2(\mu)$ has a (unique) representation

(2.10) $$f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

where

(2.11) $$\|f\|^2 = \sum_\alpha \alpha! c_\alpha^2$$

This is the Wiener-Ito chaos expansion of f (for multi-dimensional white noise).

Next we define the multi-dimensional analogue of the Albeverio-Kondratiev-Streit spaces (AKS spaces, for short) (see [AKS] or [HLØUZ 3]):

DEFINITION 2.1 [AKS]

Part a): (The stochastic test function spaces). For $0 \le \rho \le 1$ let $(\mathcal{S})^\rho$ consist of all

$$f = \sum_\alpha c_\alpha H_\alpha \in L^2(\mu)$$

such that

(2.12) $$\|f\|^2_{\rho,k} := \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho} (2\mathrm{N})^{\alpha k} < \infty \quad \text{for all } k,$$

Here

(2.13) $$(2\mathrm{N})^\alpha = \prod_{j=1}^n (2^d \beta_1^{(j)} \cdots \beta_d^{(j)})^{\alpha_j} \quad \text{if} \quad \alpha = (\alpha_1, \cdots, \alpha_n)$$

where $\beta^{(j)} = (\beta_1^{(j)}, \cdots, \beta_d^{(j)})$ is as described above.

Part b): (The stochastic distribution spaces). For $0 \le \rho \le 1$ let $(\mathcal{S})^{-\rho}$ consist of all formal expansions

$$F = \sum_\alpha b_\alpha H_\alpha$$

such that

(2.14) $$\sum b_\alpha^2 (\alpha!)^{1-\rho} (2\mathrm{N})^{-\alpha q} < \infty \quad \text{for some } q < \infty$$

The family of seminorms $\|f\|_{\rho,k}$; $k = 1, 2, \cdots$ gives rize to a topology on $(\mathcal{S})^\rho$ and we can regard $(\mathcal{S})^{-\rho}$ as the dual of $(\mathcal{S})^\rho$ by the action

(2.15) $$< F, f >= \sum_\alpha b_\alpha c_\alpha \alpha!$$

if $F=\sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})^{-\rho}$, $f=\sum_\alpha c_\alpha H_\alpha \in (\mathcal{S})^{\rho}$.

REMARK For general $\rho \in [0,1]$ we have

$$(\mathcal{S})^1 \subset (\mathcal{S})^\rho \subset (\mathcal{S})^0 \subset (\mathcal{S})^{-0} \subset (\mathcal{S})^{-\rho} \subset (\mathcal{S})^{-1}$$

Observe in particular that with this notation $(\mathcal{S})^0$ and $(\mathcal{S})^{-0}$ are different spaces. Indeed, comparing Definition 2.1 with Zhang's characterization [Z] one can see that $(\mathcal{S})^0$ and $(\mathcal{S})^{-0}$ coincide with the *Hida test function space* $(\mathcal{S})$ and the *Hida space of generalized white noise functionals* $(\mathcal{S})^*$, respectively. See [HKPS] for more information on these spaces.

We can now define the *Wick product*:

DEFINITION 2.2 The Wick product $F \diamond G$ of two elements

$$F=\sum_\alpha b_\alpha H_\alpha,\ G=\sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})^{-1}$$

is defined by

$$F \diamond G = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta}$$

The following is a consequence of Definition 2.1:

LEMMA 2.3

(i) $F, G \in (\mathcal{S})^{-1} \Rightarrow F \diamond G \in (\mathcal{S})^{-1}$
(ii) $F, G \in (\mathcal{S})^{1} \Rightarrow F \diamond G \in (\mathcal{S})^{1}$

The *Hermite transform*, introduced in [LØU 1-3], has a natural extension to $(\mathcal{S})^{-1}$:

DEFINITION 2.4 If $F=\sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})^{-1}$ then the *Hermite transform* of F, $\mathcal{H}F=\tilde{F}$, is defined by

$$\tilde{F}(z)=\mathcal{H}F(z)=\sum_\alpha b_\alpha z^\alpha \quad \text{(whenever convergent)}$$

where $z=(z_1,z_2,\cdots)\in \mathbf{C}^{\mathbf{N}}$ (the set of all finite sequences of complex numbers) and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}\cdots z_n^{\alpha_n} \quad \text{if} \quad \alpha=(\alpha_1,\cdots,\alpha_n)$$

One can show that (see [HLØUZ 3, (2.32)])

$$\tilde{F}(z_1,\cdots,z_k)=\langle F,\eta\rangle \tag{2.16}$$

where $\eta\in(\mathcal{S})^1_{\mathbf{C}}$ is the (complex) stochastic test function

$$\eta(\omega)=Exp < \omega, z_1e^{(1)}+\cdots+z_ke^{(k)} >:= \sum_{n=0}^{\infty}\frac{1}{n!} < \omega, z_1e^{(1)}+\cdots+z_ke^{(k)} >^{\diamond n}. \tag{2.17}$$

Combining Definition 2.4 with Definition 2.2 we get

LEMMA 2.5 If $F, G \in (\mathcal{S})^{-1}$ then

$$\mathcal{H}(F \diamond G)(z) = (\mathcal{H}F)(z) \cdot (\mathcal{H}G)(z) \quad \text{(when convergent)}$$

where the product on the right hand side is the usual complex product in $\mathbf{C}$.

There are several reasons why the Wick product is important:

1) The Wick product (or a related variant of it) has been used for a long time in quantum physics
2) The Wick product is a natural multiplication in $(\mathcal{S})^{-1}$ and it coincides with ordinary multiplication if one of the factors is deterministic
3) The Wick product is closely connected to stochastic integration via the formula

$$\int_0^T Y_t \delta B_t = \int_0^T Y_t \diamond W_t dt \tag{2.18}$$

where the integral on the left is the Skorohod integral (which coincides with the Ito integral if the integrand $Y_t(\omega)$ is adapted) and the integral on the right is defined as a Pettis integral in $(\mathcal{S})^*$. See e.g. [LØU 2] and see [B].

It is easily seen that if $F \in (\mathcal{S})^{-\rho}$ for some $\rho < 1$ then $\tilde{F}(z)$ converges for all finite sequences $z = (z_1, \cdots, z_n)$ of complex numbers (see e.g. [HLØUZ 3]).

For $(\mathcal{S})^{-1}$ however, the situation is different:

For $q < \infty, R < \infty$ define the "ellipsoids" $\mathbf{B}_q(R)$ by

$$\mathbf{B}_q(R) = \{(\zeta_1, \zeta_2, \cdots) \in \mathbf{C}^{\mathbf{N}}; \sum_{\alpha \neq 0} |\zeta^\alpha|^2 (2\mathbf{N})^{\alpha q} < R^2\} \tag{2.19}$$

Note that $q_1 < q_2 \Rightarrow \mathbf{B}_{q_2}(R) \subset \mathbf{B}_{q_1}(R)$.

LEMMA 2.6 [AKS] (Characterization lemma for $(\mathcal{S})^{-1}$)

a) If $F(\omega) = \sum_\alpha a_\alpha H_\alpha(\omega) \in (\mathcal{S})^{-1}$ then there exists $q < \infty, M_q < \infty$ such that

$$|\tilde{F}(z)| \leq \sum_\alpha |a_\alpha||z^\alpha| \leq M_q \cdot [\sum_\alpha (2\mathbf{N})^{\alpha q} |z^\alpha|^2]^{\frac{1}{2}} \quad \text{for all } z \in \mathbf{C}^{\mathbf{N}}$$

In particular, $\tilde{F}$ is a bounded analytic function on $\mathbf{B}_q(R)$ for all $R < \infty$.

b) Conversely, suppose $g(z) = \sum_\alpha b_\alpha z^\alpha$ is a given power series of $z \in \mathbf{C}^{\mathbf{N}}$ such that there exists $q < \infty$ and $\delta > 0$, such that ($g(z)$ is absolutely convergent when $z \in \mathbf{B}_q(\delta)$ and)

$$\sup_{z \in \mathbf{B}_q(\delta)} |g(z)| < \infty. \tag{2.20}$$

Then there exists $G \in (\mathcal{S})^{-1}$ such that $\tilde{G} = g$, namely

$$G(\omega) = \sum_\alpha b_\alpha H_\alpha(\omega).$$

Proof.

a) We have

$$|\tilde{F}(z)| \leq \sum_\alpha |a_\alpha| \cdot |z^\alpha| \leq [\sum_\alpha |a_\alpha|^2 (2\mathbf{N})^{-\alpha q}]^{\frac{1}{2}} [\sum_\alpha |z^\alpha|^2 (2\mathbf{N})^{\alpha q}]^{\frac{1}{2}}.$$

Since $F \in (\mathcal{S})^{-1}$, we see that $M_q := [\sum_\alpha |a_\alpha|^2 (2\mathbf{N})^{-\alpha q}]^{\frac{1}{2}} < \infty$ if q is large enough.

b) Conversely, assume that (2.20) holds. Then

$$\sum_\alpha |b_\alpha| \cdot |z^\alpha| < \infty \text{ for all } z \in \mathbf{B}_q(\delta), \text{ for all } q \geq q_0, \text{ say} \tag{2.21}$$

For $r < \infty$ choose $z = z^{(r)} = (z_1, z_2, \cdots)$ with

$$z_j = (2^d \beta_1^{(j)} \cdots \beta_d^{(j)})^{-r}\ ;\ j = 1, 2, \cdots$$

Then

$$\sum_\alpha (2\mathbf{N})^{\alpha r} |z^\alpha|^2 = \sum_\alpha (2\mathbf{N})^{-\alpha r} < \delta^2$$

is r is large enough,say $r \geq q_1$. Hence $z \in \mathbf{B}_r(\delta)$ for $r \geq q_1$ so by (2.21) we have, for $q \geq\ max\ (q_0, q_1)$,

$$\sum_\alpha |b_\alpha| (2\mathbf{N})^{-\alpha q} = \sum_\alpha |b_\alpha| z^\alpha \leq \sum_\alpha |b_\alpha| \cdot |z^\alpha| < \infty.$$

In particular,

$$K := \sup_\alpha |b_\alpha| (2\mathbf{N})^{-\alpha q} < \infty.$$

This gives

$$\sum_\alpha |b_\alpha|^2 (2\mathbf{N})^{-2\alpha q} \leq K \cdot \sum_\alpha |b_\alpha| (2\mathbf{N})^{-\alpha q} < \infty,$$

so $G := \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})^{-1}$ as claimed.

EXAMPLE 2.7

Choose $F = W_\phi^{(i)}$, the i'th component of m-dimensional white noise, $\phi = (\phi_1, \cdots, \phi_m) \in \mathcal{S}$. Then $F(\omega) =< \omega_i, \phi_i >$ if $\omega = (\omega_1, \cdots, \omega_m) \in \mathcal{S}$. Since $\phi_i \in L^2(\mathbf{R}^d)$ we can write

$$\phi_i = \sum_j \lambda_{ij} \eta_j$$

and then we get

$$\begin{aligned} W_\phi^{(i)}(\omega) &= \sum_j \lambda_{ij} < \omega_i, \eta_j >= \sum_j \lambda_{ij} < \omega, e_{m(j-1)+i} >= \sum_j \lambda_{ij} h_1(< \omega, e_{m(j-1)+i} >) \\ &= \sum_j \lambda_{ij} H_{\epsilon_{m(j-1)+i}}(\omega), \end{aligned}$$

where $\epsilon_k = (0, 0, \cdots, 0, 1, 0, \cdots)$ with 1 on k'th place.

So the Hermite transform of $W_\phi^{(i)}$ is

$$\mathcal{H}W_\phi^{(i)}(z_1, z_2, \cdots) = \sum_j \lambda_{ij} z_{m(j-1)+i} \tag{2.22}$$

We see that the Hermite transforms of the components $W_\phi^{(i)}$ of $\vec{W}_\phi$ involve disjoint families of z_k-variables.

EXAMPLE 2.8 The white noise $W_\phi^{(i)}$ presented so far may be regarded as a ϕ-smoothed version of a *singular* white noise $W_x^{(i)} \in (\mathcal{S})^{-1}$ defined by

$$W_x^{(i)}(\omega) = \sum_j \eta_j(x) H_{\epsilon_{m(j-1)+i}}(\omega) \quad ; x \in \mathbf{R}^d \tag{2.23}$$

The $\mathcal{H}$-transform of $W_x^{(i)}$ is

$$\mathcal{H}W_x^{(i)}(z) = \sum_j \eta_j(x) z_{m(j-1)+i} \quad ; \quad z = (z_1, z_2, \cdots) \tag{2.24}$$

The m-dimensional singular white noise is denoted by $\vec{W}_x(\omega) = (W_x^{(1)}(\omega), \cdots, W_x^{(m)}(\omega))$.

Note that it follows from Lemma 2.6b) that the analytic functions operate on the $\mathcal{H}$-transforms of $(\mathcal{S})^{-1}$ (see also [GHLØUZ]).

LEMMA 2.9 [AKS] Suppose $g = \mathcal{H}X$ for some $X \in (\mathcal{S})^{-1}$. Let $f : V \subset \mathbf{C} \to \mathbf{C}$ be analytic, where V is a neighbourhood of $\zeta_0 = E[X] = g(0)$ in $\mathbf{C}$. Then there exists $Y \in (\mathcal{S})^{-1}$ such that

$$\mathcal{H}Y = f \circ g$$

Finally we mention another important result based on [AKS] and Lemma 2.6:

THEOREM 2.10 [AKS]

The following are equivalent:

(i) $X_n \to X$ in $(\mathcal{S})^{-1}$
(ii) $\exists \delta > 0, q < \infty$ such that

$$\tilde{X}_n(z) \to \tilde{X}(z) \quad \text{pointwise boundedly in } \mathbf{B}_q(\delta)$$

A model for stochastic permeability

Following the approach of [Gj] we now apply the above to construct a model for the stochastic permeability matrix $K(x, \omega)$. As mentioned earlier $K(x, \omega)$ should be a symmetric, non-negative definite $d \times d$-matrix for each x and ω. Moreover, it is usually assumed that K has, at least approximately, the following probabilistic features:

(2.25) (Independence) If $x_1 \neq x_2$ then $K(x_1, \cdot)$ and $K(x_2, \cdot)$ are independent

(2.26) (Lognormality) For each x the eigenvalues $\lambda_1(x, \cdot), \cdots, \lambda_d(x, \cdot)$ of $K(x, \cdot)$ have a lognormal distribution

(2.27) (Stationarity) For all $x_1, \cdots, x_n \in \mathbf{R}^d$ and $h \in \mathbf{R}^d$ the random variable

$$Y = (K(x_1 + h, \cdot), \cdots, K(x_n + h, \cdot))$$

has a distribution which is independent of h.

A natural candidate for such a $K(x, \omega)$ is constructed as follows:

Let $\mathcal{W}_x(\omega)$ be the stochastic $d \times d$ matrix

$$\mathcal{W}_x(\omega) = \begin{bmatrix} W_x^{1,1}(\omega) & \cdots & W_x^{1,d}(\omega) \\ \vdots & & \vdots \\ W_x^{d,1}(\omega) & \cdots & W_x^{d,d}(\omega) \end{bmatrix}, \tag{2.28}$$

where $W_x^{i,j}(\omega) = W_x^{j,i}(\omega)$ for all i, j, x, ω and $\vec{W}_x := (W^{i,j})_{i \leq j}$ is $\frac{1}{2}d(d+1)$-dimensional white noise. $\mathcal{W}_x(\omega)$ is called the *symmetric white noise matrix.*

Let $\mathcal{W}_x^{\diamond n}$ denote the *Wick matrix power of order* n, so that

$$\mathcal{W}^{\diamond 2} = \mathcal{W} \diamond \mathcal{W} = [\sum_{j=1}^{d} W^{i,j} \diamond W^{j,k}]_{i,k} \quad \text{and so on.} \tag{2.29}$$

Then define

$$K(x, \omega) := Exp \mathcal{W}_x(\omega) := \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{W}_x^{\diamond n}(\omega) \tag{2.30}$$

Just as for white noise itself there is a *smoothed* variant $K_\phi(\omega)$ of the singular stochastic permeability matrix $K(x, \omega)$. The smoothed variant is defined by

$$K_\phi(\omega) = Exp \mathcal{W}_\phi(\omega) \text{ where } \mathcal{W}_\phi(\omega) = \begin{bmatrix} W_\phi^{1,1}(\omega) & \cdots & W_\phi^{1,d}(\omega) \\ \vdots & & \vdots \\ W_\phi^{d,1}(\omega) & \cdots & W_\phi^{d,d}(\omega) \end{bmatrix} \tag{2.31}$$

The smoothing is more than a technical convenience: The ideal requirement of independence, (2.25), is unrealistic. However, if we put

$$K_\phi(x, \omega) := K_{\phi_x}(\omega) \tag{2.32}$$

where

$$\phi_x(y) = \phi(y - x) \tag{2.33}$$

is the x-shift of the test function (or "window") ϕ, we see that K_ϕ satisfies (2.26), (2.27) and

(2.25)' (Weak independence) If supp$\phi_{x_1} \cap$ supp$\phi_{x_2} = \phi$, then $K_\phi(x_1, \cdot)$ and $K_\phi(x_2, \cdot)$ are independent.

Smoothing by ϕ_x represents taking the macroscopic ϕ-average shifted to x and in a physical application suppϕ should be chosen large compared to the pores of the medium but small compared to the macroscopic properties of the flow. See the discussion in [HLØUZ 3], [LØU 1].

If K_ϕ is given by (2.31) then clearly $K_\phi(x, \omega)$ is symmetric for each x, ω. By Prop. 1.31 in [G] $K_\phi(x, \omega)$ is positive definite and the eigenvalues $\lambda_i(x, \omega)$ of $K_\phi(x, \omega)$ can be written on the form

$$\lambda_i(x, \omega) = (\xi^{(i)})^T K_\phi(x, \omega)\xi^{(i)} = ExpW_\psi(\omega)$$

for suitable choice of $\xi^{(i)} \in \mathbf{R}^d$, $|\xi^{(i)}| = 1$, and $\psi = \psi(x, \phi, \xi^{(i)}) \in \mathbf{R}^d$ (T denotes transposed) where $W_\psi(\omega)$ is 1-dimensional white noise.

Thus we see that $K_\phi(x, \omega)$ given by (2.31) does indeed satisfy the requirements (2.25)', (2.26) and (2.27) and from now on we will use this as a model for the stochastic permeability matrix of a porous anisotropic medium.

CHAPTER 3. A GENERAL SCHEME FOR SOLVING STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS INVOLVING FUNCTIONALS OF WHITE NOISE

The results established in Chapter 2 can be applied to a wide class of stochastic partial (or ordinary) differential equations, not just those associated to fluid flow in porous media. The general procedure is the following:

STEP 1: Assume that the "noise" in the equation can be modelled as some (Wick) functional of (possibly multidimensional) white noise and interpret all products/functions involved as Wick products/Wick functions. (This corresponds to a Skorohod interpretation of the stochastic integrals involved See e.g. [LØU 1-3], [B])

STEP 2: Apply the Hermite transform $\mathcal{H}$ to the equation. This turns the original stochastic equation involving the unknown random field $x \to X(x, \omega)$ into a *deterministic* partial differential equation (with usual products) with the unknown $x \to v(x, z) := \tilde{X}(x, z)$, where $z = (z_1, z_2, \cdots) \in \mathbf{C}^{\mathbf{N}}$ is a complex parameter.

STEP 3: Solve this deterministic partial differential equation, for all $z \in \mathbf{B}_q(0, R)$ with $q < \infty$ sufficiently large and all $R < \infty$.

STEP 4: Verify that the solution $v(x, z)$ is a bounded analytic function on $\mathbf{B}_q(0, R)$ for all $R < \infty$. Then by the Characterization Lemma (2.5) $v(x, z)$ is indeed the Hermite transform of an element $X(x, \cdot) \in (\mathcal{S})^{-1}$, which then solves the original equation.

We illustrate this procedure on some examples:

EXAMPLE 3.1

A commonly used model for exponential growth under uncertainty of m correlated quantities $X_t^{(1)}(\omega),\ldots,X_t^{(m)}$ is

$$\frac{dX_t^{(i)}(\omega)}{dt} = (\alpha_i(t) + \sum_{j=1}^{n} \beta_{ij}(t)\cdot W_t^{(j)}(\omega))\cdot X_t^{(i)}(\omega);\ 1\le i\le n \tag{3.1}$$

where $\vec{W}_t(\omega) = (W_t^{(1)}(\omega),\cdots,W_t^{(n)}(\omega))$ is n-dimensional white noise and $\alpha_i(t),\beta_{ij}(t)$ are bounded deterministic functions. In view of (2.18) we interpret the product in (3.1) as a Wick product:

$$\frac{dX_t^{(i)}}{dt} = (\alpha_i + \sum_{j=1}^{n} \beta_{ij}W_t^{(j)})\diamond X_t^{(i)}\ ;\ 1\le i\le n \tag{3.2}$$

Taking $\mathcal{H}$-transform we get the following equations for $v^{(i)}(t,z) = \tilde{X}_t^{(i)}(z)$:

$$\frac{dv^{(i)}(t,z)}{dt} = (\alpha_i + \sum_{j=1}^{n} \beta_{ij}\tilde{W}_t^{(j)})v^{(i)}(t,z)\ ;\ 1\le i\le n \tag{3.3}$$

These equations have the solution

$$v^{(i)}(t,z) = v^{(i)}(0,z)\cdot\exp(\int_0^t \alpha_i(s)ds + \sum_{j=1}^{n}\int_0^t \beta_{ij}(s)\tilde{W}_s^{(j)}(\omega)ds) \tag{3.4}$$

Taking inverse $\mathcal{H}$-transform we end up with the solution

$$X_t^{(i)}(\omega) = X_0^{(i)}(\omega)\diamond Exp[\int_0^t \alpha_i(s)ds + \sum_{j=1}^{n}\int_0^t \beta_{ij}(s)dB_j(s)];1\le i\le n \tag{3.5}$$

Note that this is valid regardless of whether the initial values $X_0^{(i)}(\omega)$ are adapted or not. If $X_0^{(i)}(\omega) = x_i$ is deterministic, the solution coincides with the solution obtained by ordinary Ito calculus, because

$$Exp(\int_0^t \beta_{ij}(s)dB_s) = \exp(\int_0^t \beta_{ij}(s)dB - \frac{1}{2}\int_0^t \beta_{ij}^2(s)ds) \tag{3.6}$$

(see e.g. [GHLØUZ]).

REMARK. If we replace the singular white noise $\vec{W}_t(\omega)$ by the smoothed variant $\vec{W}_\phi(t,\omega)$ (and keep the Wick product) in the example above, we get the solution $X_t^{(\phi,i)}(\omega)$ given by

$$(3.7) \qquad X_t^{(\phi,i)}(\omega) = X_0^{(i)}(\omega) \diamond Exp[\int_0^t \alpha_i(s)ds + \sum_{j=1}^{n} \int_0^t \beta_{ij}(s) W_\phi^{(j)}(s,\omega)ds] \; ; \; 1 \le i \le n$$

In this case it is easily seen that $X_t^{(\phi,i)}(\omega) \to X_t^{(i)}(\omega)$ in $L^2(\mu)$ as $\phi \to \delta_0$ (the Dirac measure at 0). Bur for many stochastic partial differential equations this limit will not exist, at least not in $L^p(\mu)$. Examples of such equations are given in Chapter 4. On the other hand, such equations make perfectly good sense in the ϕ-smoothed case and there is no physical reason that the limit should exist as ϕ approaches the Dirac measure.

CHAPTER 4. APPLICATIONS TO HYDRODYNAMICS

4a) The pressure equation for one phase fluid flow in a stochastic medium.

Finally we return to the equations for fluid flow in a porous medium discussed in Chapter 1, but now for the stochastic case. We represent the stochastic permeability K by the Wick exponential of the symmetric smoothed white noise matrix, as suggested by (2.25):

$$(4.1) \qquad K = K_\phi(x,\omega) = Exp\mathcal{W}_{\phi_x}(\omega) \quad ; \; \phi \in \mathcal{S}$$

In view of the discussion in Chapter 2 we interpret all products in the relevant equations as Wick products. We first consider the stochastic version of the moving boundary problem (1.14)-(1.16). We will here not discuss the general case but only consider the situation for one fixed time $t \geq 0$, a given bounded open set $D_t = D$ and a given source rate $\xi_t = \xi$. Then the problem is to find $p(x,\omega)$ such that

$$(4.2) \qquad div(K_\phi(x,\cdot) \diamond \nabla p(x,\cdot)) = -\xi(x) \; ; \; x \in D$$

and

$$(4.3) \qquad p(x,\cdot) = 0 \; ; \; x \in \partial D$$

The precise meaning of (4.2) is that $p(x,\cdot) \in (\mathcal{S})^{-1}$ for each x and that for each *stochastic* test function $\eta \in (\mathcal{S})^1$ we have

$$(4.4) \qquad \langle \text{div}(K_\phi(x,\cdot) \diamond \nabla p(x,\cdot)), \eta \rangle = -\xi(x) E[\eta]; \quad x \in D$$

Since the exponential test functions

$$(4.5) \qquad \eta(\omega) = \text{Exp}\langle \omega, \varphi \rangle; \quad \omega \in \mathcal{S}', \; \varphi \in \mathcal{S}$$

are dense in $(\mathcal{S})^{-1}$, we see by (2.16) and Lemma 2.5 that (4.4) is equivalent to

$$(4.6) \qquad \text{div}(\tilde{K}_\phi(x,z) \cdot \nabla \tilde{p}(x,z)) = -\xi(x); \; x \in D, \; z \in \mathbf{C}_0^{\mathbf{N}}$$

In the case when the medium is *isotropic* we have

$$K_\phi(x,\omega) = k_\phi(x,\omega)\cdot I = \mathrm{Exp}W_\phi(x,\omega)\cdot I \tag{4.7}$$

where I is the $d\times d$ identity matrix and W_ϕ is 1-dimensional white noise. In this case it is shown in [HLØUZ 3] that (4.2)-(4.3) has a unique solution $p(x,\cdot)\in(\mathcal{S})^{-1}$. This solution is expressed in terms of an auxiliary Brownian motion $b_t(\hat\omega)$; $\hat\omega\in\hat\Omega$ in $\mathbf{R}^d$ as follows:

$$p(x,\omega) = \frac{1}{2}\mathrm{Exp}(-\frac{1}{2}W_\phi(\omega))\diamond \hat{E}^x\Big[\int_0^\tau \xi(b_t(\hat\omega))\mathcal{G}(t,b_t(\hat\omega),\omega)dt\Big] \tag{4.8}$$

where

$$\mathcal{G}(t,x,\omega) = \mathrm{Exp}\{\frac{1}{2}W_{\phi_x}(\omega) - \frac{1}{4}\int_0^t\Big[\frac{1}{2}(\nabla W_{\phi_x})^{\diamond 2}(\omega) + \Delta W_{\phi_x}(\omega)\Big]ds\}, \tag{4.9}$$

$\tau=\tau(\hat\omega)=\inf\{t>0; b_t(\hat\omega)\notin D\}$ and $\hat{E}^x$ denotes expectation with the respect to the law of $b_t(\hat\omega)$ starting at x (the gradient and the Laplacian are both taken with respect to x).

4b) Two phase fluid flow in a stochastic isotropic medium

If the medium is stochastic but isotropic then the following model for two phase fluid flow has been suggested by [Gj]:

We first consider the deterministic equations, assuming for simplicity that both fluids have viscosity and density equal to 1:

(4.10) Darcy's law for two phase flow:

$$\vec{q}_1 = -k_{11}\nabla p_1 - k_{12}\nabla p_2$$
$$\vec{q}_2 = -k_{21}\nabla p_1 - k_{22}\nabla p_2$$

where $\vec{q}_i(t,x)$ is the seepage velocity of fluid number i, $p_i(t,x)$ is the pressure of fluid number i and $k_{ij}(x)$ is the relative permeability with respect to fluid number i and j. It is usually assumed (and it can be proved under more basic assumptions) that $k_{ij}=k_{ji}$.

(4.11) The continuity equations:

$$\frac{\partial\theta_i}{\partial t} = -div(\vec{q}_i) + \xi_i;\quad i=1,2$$

where $\theta_i(t,x)$ is the saturation of fluid number i and $\xi_i(t,x)$ is the source rate of fluid number i.

Combining (4.10)-(4.11) we get

$$\frac{\partial \theta_i}{\partial t} = \sum_{j=1}^{2} div(k_{ij} \nabla p_j) + \xi_i; \quad i = 1, 2 \tag{4.12}$$

As in the one phase situation let us now assume that at each time the saturation $\theta_i(t,x)$ is either 0 or has a maximal value $\tilde{\theta}_i(x) > 0$.

Define

$$D_t^i = \{x; \theta_i(t,x) = \tilde{\theta}_i(x)\}; \quad i = 1, 2 \quad \text{(the wet region for fluid } i\text{)} \tag{4.13}$$

If we assume that the fluids are immiscible we have

$$D_t^1 \cap D_t^2 = \emptyset \quad \text{for all } t \geq 0 \tag{4.14}$$

Now we can proceed as before in Chapter 1. By the same argument as in Chapter 2 we use the representation

$$k_{ij}(x,\omega) = \text{Exp} W_{ij}^{\phi_x}(\omega); \quad k_{12} = k_{21}$$

where $\tilde{W} = (W_{11}, W_{12}, W_{22})$ is 3-dimensional white noise. Then we end up with the following system of stochastic moving boundary problems ([Gj, §3.2])

$$\text{div}(k_{ij} \diamond \nabla p_j) = -\delta_{ij} \xi_j \mathcal{X}_{D_t^j}; \quad i, j = 1, 2 \tag{4.15}$$

$$p_j(t, x, \cdot) = 0 \quad \text{for } x \in D_t^j \tag{4.16}$$

$$\tilde{\theta}_i \cdot \frac{d}{dt}(\partial D_t^i) = -\sum_{j=1}^{2} k_{ij} \nabla p_j \quad \text{for } x \in \partial D_t^i; \ i = 1, 2. \tag{4.17}$$

4c) The Burgers equation with a stochastic force

Next we turn to an important special case of the first order nonlinear saturation equation (1.6), namely the *Burgers equation*

$$\frac{\partial u_k}{\partial t} + \lambda \sum_{j=1}^{n} u_j \frac{\partial u_k}{\partial x_j} = \nu \Delta u_k \tag{4.18}$$

where $(t,x) \in \mathbf{R}^{d+1}$ and λ, ν are constants, $\nu > 0$.

In addition to fluid flow this equation has several applications. For example, it appears in the study of the growth of interfaces, e.g. the way solids form through growth processes on the surface.

A stochastic variant of this equation based on Wick products has been studied in [HLØUZ 2]:

$$\frac{\partial u_k}{\partial t} + \lambda \sum_{j=1}^{d} u_j \diamond \frac{\partial u_k}{\partial x_j} = \nu \Delta u_k + w_k \tag{4.19}$$

where $w = (w_1, \ldots, w_d)$ is d-dimensional noise, interpreted as a stochastic force. It is shown that under certain conditions the equation (4.19) can be transformed into a stochastic heat equation, by a Wick version of the Cole-Hopf transformation. This stochastic heat equation can be solved using the procedure outlined in Chapter 3. We refer to [HLØUZ 2] for details. The stochastic Burgers equation (4.19) with $d = 1$ and ordinary product instead of Wick product has been studied in [BCJ-L]. See also [HR].

4d) The transport equation in a turbulent medium

Consider a substance dissolved in a moving fluid in $\mathbf{R}^d$, being exposed to both a molecular diffusion and to a drift coming from the movement of the fluid. If the fluid is turbulent, a natural model for its velocity is d-dimentional white noise $\vec{W}_{\varphi_x}$. The consentration $u(t, x, \omega)$ of the substance at (t, x) will then satisfy the stochastic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\nu^2 \Delta u + \vec{W}_{\varphi_x} \diamond \nabla u \tag{4.20}$$

where $\nu > 0$ is the molecular viscosity of the fluid.

In [GjHØUZ] this equation is solved given the initial values

$$u(0, x, \omega) = f(x, \omega) \tag{4.21}$$

under certain smoothness conditions on the given function f (f may be anticipating).

With Wick product replaced by usual product this equation has been studied by several authors. See e.g. [Po] and also the references there.

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